On the algebraic dual of $\mathcal{D}(\Omega)$

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Abstract

This paper is concerned with the algebraic dual $\mathcal{D}^*(\Omega)$ of the space of test functions $\mathcal{D}(\Omega)$. The emphasis is on failures and successes of $\mathcal{D}^*(\Omega)$ as compared to the continuous dual $\mathcal{D}'(\Omega)$, the space of distributions. Topological properties, operations with elements of $\mathcal{D}^*(\Omega)$ and applications to linear partial differential equations are discussed.

Keywords

Algebraic dual, distributions, partial differential equations, locally convex spaces.

1 Introduction

Topology plays a prominent and indispensable role in the theory of distributions, as has been emphasized e.g. by John Horváth in his monograph [5] as well as at numerous other places [6, 7, 8, 11, 12, 13, 14]. For example, the space of distributions $\mathcal{D}'(\Omega)$ on an open subset Ω of $\mathbb{R}^n$ is defined as the continuous dual of the space of test functions $\mathcal{D}(\Omega)$; similarly, all other spaces of distributions can be viewed as continuous duals. The fact that the elements of $\mathcal{D}'(\Omega)$ are *continuous* linear functionals is essential in many constructions as well as applications to partial differential equations. When teaching distribution theory one usually has to spend some effort on explain-

ing the topology of $\mathcal{D}(\Omega)$ as a locally convex inductive limit of Fréchet spaces. Thus occasionally the question arises what would happen if one dropped continuity from the definition of $\mathcal{D}'(\Omega)$ and considered the algebraic dual $\mathcal{D}^*(\Omega)$ instead. In this paper I wish to pursue this question, and in particular, show what goes wrong with $\mathcal{D}^*(\Omega)$ at the hand of a number of examples.

While these examples will clearly exhibit the lack of certain desirable properties of $\mathcal{D}^*(\Omega)$ for the purpose of analysis, it is curious to note that as a topological vector space, $\mathcal{D}^*(\Omega)$ has better properties than $\mathcal{D}'(\Omega)$. Not surprisingly, certain partial differential equations that do not have solutions in $\mathcal{D}'(\Omega)$ turn out to be solvable in $\mathcal{D}^*(\Omega)$. For example, constant coefficient partial differential operators have solutions in $\mathcal{D}^*(\Omega)$ on *every* open subset of $\mathbb{R}^n$ with arbitrary members of $\mathcal{D}^*(\Omega)$ on the right hand side. A similar solvability result in $\mathcal{D}^*(\Omega)$ will be seen to hold, e. g., for the Lewy equation. This, however, is counterbalanced by the fact that one cannot say much about the behavior of these solutions, having lost control over their analytical properties due to arguments involving algebraic bases.

The plan of the paper is as follows. In Section 2 some basic notions needed in the sequel are recalled. In Section 3 properties of $\mathcal{D}^*(\Omega)$ as a topological vector space are collected. Although these results are known it seemed appropriate to arrange them in the context of the theme of the paper. In Section 4, I present a number of assertions and examples demonstrating failures (and successes) of $\mathcal{D}^*(\Omega)$. To my knowledge, these considerations have not appeared in print so far. On the positive side, we will see that derivation, multiplication by smooth functions and sheaf theoretic arguments work well in $\mathcal{D}^*(\Omega)$. On the negative side, we will encounter the failure of convolution to regularize, difficulties with the definition of tensor products and convolution, and the lack of the notion of local order in $\mathcal{D}^*(\Omega)$. Finally, Section 5 contains some observations on solvability of partial differential equations. We dwell a bit on the role of P-convexity, hypoellipticity and fundamental solutions in $\mathcal{D}^*(\Omega)$ there (part of the latter results are based on joint work with T. Todorov [22]).

2 Notation

Throughout the paper, I follow the notation of [5]. The term *locally convex space* will refer to a locally convex Hausdorff topological vector space over

the field $\mathbb{K}$ ($\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$ in the sequel). If the vector spaces F, G form a dual system (F, G) [5, Def. 3.2.1], the *weak-, Mackey-* and *strong topologies* on F are denoted by $\sigma(F,G), \tau(F,G)$ and $\beta(F,G)$, respectively. These are the topologies of uniform convergence on the finite subsets of G, on the absolutely convex, $\sigma(G,F)$-compact subsets of G, and on the $\sigma(G,F)$-bounded subsets of G, respectively. The *algebraic dual* of a vector space E is the set of all linear maps from E into $\mathbb{K}$ and will be denoted by E^*. If E is a locally convex space with topology $\mathcal{T}$, the *continuous dual* or simply *dual* is the set of linear forms continuous with respect to the topology $\mathcal{T}$ and will be denoted by E'. It is known that E' is the dual of E with respect to every locally convex topology finer than $\sigma(E,E')$ and coarser than $\tau(E,E')$ [5, Prop. 3.5.4]. A locally convex space E is *complete* if every Cauchy filter on E converges. An absolutely convex, absorbing and closed subset of F is called a barrel. The locally convex space E is called *barrelled*, if every barrel is a neighborhood of zero. The family of *all* absolutely convex, absorbing subsets of a vector space E generates the finest (i. e., largest) locally convex topology on E [5, Ex. 2.4.3], which we denote by $\mathcal{T}_\ell$.

Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{E}(\Omega)$ is the space of infinitely differentiable functions on Ω with values in $\mathbb{K} = \mathbb{C}$. Equipped with the topology of uniform convergence on compact subsets of Ω, it is a complete and metrizable locally convex space (a *Fréchet* space). The support of a smooth function is the closure (in Ω) of the set of points on which it does not vanish. Given a compact subset $K \subset \Omega$, $\mathcal{D}_K(\Omega)$ denotes the subspace of $\mathcal{E}(\Omega)$ of smooth functions with support in K. The union of all $\mathcal{D}_K(\Omega)$ as K runs through the compact subsets of Ω is the space $\mathcal{D}(\Omega)$ of *compactly supported* smooth functions. Its genuine topology $\mathcal{T}_\mathcal{D}$ is the final locally convex topology with respect to all injections $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$, with which it is a strict inductive limit of Fréchet spaces [5, Sect. 2.12]. The space of distributions on Ω, $\mathcal{D}'(\Omega)$, is the continuous dual of $\mathcal{D}(\Omega)$. Given $S \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, the action of S on φ is denoted by $\langle S, \varphi \rangle$. If U is an open subset of Ω, there is a natural injection of $\mathcal{D}(U)$ into $\mathcal{D}(\Omega)$; its transpose defines the restriction map of $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(U)$. The support of a distribution $S \in \mathcal{D}'(\Omega)$ is the complement of the largest open set U such that the restriction of S to U vanishes.

The space $\mathcal{D}(\Omega)$ is densely imbedded in $\mathcal{E}(\Omega)$; hence the transpose of the imbedding is injective - this way $\mathcal{E}'(\Omega)$ can be viewed as a subspace of $\mathcal{D}'(\Omega)$ and in fact be identified with the space of distributions with compact support [5, Prop. 4.2.3]. Recall also that any locally integrable function f can be

viewed as a distribution, given by the action $\varphi \to \int f(x)\varphi(x)\,\mathrm{d}x$ for $\varphi \in \mathcal{D}(\Omega)$. In particular, the space of smooth functions $\mathcal{E}(\Omega)$ is contained in $\mathcal{D}'(\Omega)$.

Let K be a compact subset of $\Omega, L > 0, \sigma > 1$. The space $\mathcal{D}_\sigma(\Omega, K, L)$ is defined as the subspace of $\mathcal{D}_K(\Omega)$ of functions whose p-th partial derivatives are bounded, uniformly on Ω, by a constant times $L^{|p|}(|p|!)^\sigma$. The inductive limit of the spaces $\mathcal{D}_\sigma(\Omega, K, L)$ as K runs through all compact subsets of Ω and $L \to \infty$ is the Gevrey class of order σ, $\mathcal{D}_\sigma(\Omega)$. Its continuous dual is the space of Gevrey ultradistributions of order σ, $\mathcal{D}'_\sigma(\Omega)$, see e. g. [19, Chap. 7, Def. 2.1].

3 Topological properties of $\mathcal{D}^*(\Omega)$

General properties of algebraic duals. All results in this section are known, but will be useful and relevant for a proper understanding of $\mathcal{D}^*(\Omega)$. We begin by collecting some properties that hold for algebraic duals in general. Thus let E be a locally convex space and let $(e_\lambda)_{\lambda\in\Lambda}$ be an algebraic basis of E. Then E is algebraically isomorphic with the direct sum of $|\Lambda|$ copies of $\mathbb{K}$ and E^* with the corresponding direct product:

$$E \approx \mathbb{K}^{(\Lambda)}, \quad E^* \approx \mathbb{K}^{\Lambda}. \tag{1}$$

The space $\mathbb{K}^{(\Lambda)}$ is equipped with the finest locally convex topology making all injections $\mathbb{K}^I \to \mathbb{K}^{(\Lambda)}$, I finite, continuous. It is clear that this topology coincides with the finest locally convex topology on $\mathbb{K}^{(\Lambda)}$. Further, the product topology on $\mathbb{K}^\Lambda$ coincides with the weak topology $\sigma(\mathbb{K}^\Lambda, \mathbb{K}^{(\Lambda)})$ [5, Prop. 3.14.3]. Clearly, the dual of E with respect to the finest locally convex topology $\mathcal{T}_\ell$ is E^*. Thus, if we put the finest locally convex topology $\mathcal{T}_\ell$ on E and the weak topology $\sigma(E^*, E)$ on E^*, the isomorphisms in (1) are topological.

Lemma 1 *Let E be a vector space. Then:*
(a) Every $\sigma(E, E^*)$-bounded subset of E is finite dimensional.
(b) Every subspace of E is closed with respect to the topology $\sigma(E, E^*)$.

Proof: (a) If $(x_n)_{n\in\mathbb{N}}$ is an infinite sequence of linearly independent members of E, one can find an element $x^* \in E^*$ such that $\langle x_n, x^*\rangle = n$; thus the set $(x_n)_{n\in\mathbb{N}}$ is unbounded. (b) If L is a subspace of E and $x \notin L$, one can find a linear form which vanishes on L and has value 1 on x, say. $\square$

The topology $\sigma(E^*, E)$ is the topology of uniform convergence on the $\sigma(E, E^*)$-bounded, finite-dimensional subsets of E [5, Ex. 3.4.1], while the topology $\beta(E^*, E)$ is the topology of uniform convergence on the $\sigma(E, E^*)$-bounded subsets of E. The Mackey topology $\tau(E^*, E)$ is the topology of uniform convergence on the absolutely convex, $\sigma(E, E^*)$-compact subsets of E. Since these are $\sigma(E, E^*)$-bounded as well, Lemma 1 (a) implies that the weak-, Mackey- and strong topology coincide on E^*:

$$\sigma(E^*, E) = \tau(E^*, E) = \beta(E^*, E).$$

As noted above, E^* is the dual of E with respect to the finest locally convex topology $\mathcal{T}_\ell$. It follows from [5, Prop. 3.5.4] that

$$\mathcal{T}_\ell = \tau(E, E^*).$$

Proposition 2 *Let E be a vector space. Then:*
(a) E^* is complete with respect to $\sigma(E^*, E)$.
(b) E is complete with respect to $\tau(E, E^*)$.

Proof: This follows from the isomorphisms (1) above and [5, Rem. 2.11.1] [15, §18.5.(3)]. □

Properties related to the Banach-Steinhaus theorem. One of the important theorems of analysis is the Banach-Steinhaus theorem. In one of its forms, it relates equicontinuity and pointwise boundedness of continuous linear maps. Thus let F, G be locally convex spaces and consider the statement

(S1) Every pointwise bounded family of continuous linear maps from F into G is equicontinuous.

The question about the maximal class of locally convex spaces F such that (S1) holds for all locally convex spaces G is answered by the *Banach-Steinhaus theorem*; it is the class of barrelled locally convex spaces: *A locally convex space F satisfies (S1) for every locally convex space G if and only if it satisfies (S1) for $G = \mathbb{K}$, if and only if it is barrelled*, see e. g. [5, Prop. 3.6.2] or [21, Thm. 3.2.3]. For applications, the following corollary (see [5, Cor. to Prop. 3.6.5]) of the Banach-Steinhaus theorem is important:

Proposition 3 *Let F be a barrelled locally convex space, $(x_n^*)_{n\in\mathbb{N}}$ a sequence of continuous linear forms on F which converges pointwise, that is, $\langle x_n^*, x\rangle$ converges to a limit $\langle x^*, x\rangle$ for every $x \in F$. Then x^* defines a continuous linear form on F.* □

Corollary 4 *Let E be a vector space. Then:*
(a) E^* is barrelled with respect to the topology $\sigma(E^*, E)$.
(b) E is barrelled with respect to the finest locally convex topology $\tau(E, E^*)$.

Proof: (a) Let X be a family of pointwise bounded $\sigma(E^*, E)$-continuous linear maps from E^* into $\mathbb{K}$. By [5, Prop. 3.2.2], X is a subset of E. By Lemma 1, X is finite dimensional. Being bounded, it is also contained in the convex hull of finitely many points, hence equicontinuous. Thus property (S1) holds for $F = E^*$ and $G = \mathbb{K}$, so E^* is barrelled. (b) In the finest locally convex topology, every barrel is a neighborhood. □

Corollary 5 *If E is a barrelled locally convex space, then E' is sequentially complete with respect to $\sigma(E', E)$.*

Proof: This is an immediate consequence of Proposition 3. □

We now summarize these observations in the situation of the algebraic dual of the space of test functions and combine them with classical facts about distribution spaces. Let Ω be an open subset of $\mathbb{R}^n$. For simplicity of notation, we drop reference to the open set Ω in the expressions for the polar topologies. Thus $\tau(\mathcal{D}, \mathcal{D}')$ will mean $\tau(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$ in the remainder of this section, and similarly for the other spaces and polar topologies.

Clearly, the inductive limit topology $\mathcal{T}_{\mathcal{D}}$ on $\mathcal{D}(\Omega)$ coincides with the Mackey topology $\tau(\mathcal{D}, \mathcal{D}')$, which in turn is strictly coarser than the topology $\mathcal{T}_\ell = \tau(\mathcal{D}, \mathcal{D}^*)$.

Proposition 6 *(a) The space of test functions $\mathcal{D}(\Omega)$ is complete and barrelled with respect to both topologies $\tau(\mathcal{D}, \mathcal{D}')$ and $\tau(\mathcal{D}, \mathcal{D}^*)$.*
(b) The continuous dual $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ (with respect to $\mathcal{T}_{\mathcal{D}}$) is complete and barrelled with the topology $\beta(\mathcal{D}', \mathcal{D})$ and sequentially complete with the topology $\sigma(\mathcal{D}', \mathcal{D})$.
(c) The algebraic dual $\mathcal{D}^*(\Omega)$ of $\mathcal{D}(\Omega)$ is complete and barrelled with the topology $\sigma(\mathcal{D}^*, \mathcal{D}) = \tau(\mathcal{D}^*, \mathcal{D}) = \beta(\mathcal{D}^*, \mathcal{D})$.

(d) $\mathcal{E}(\Omega)$ is dense, but not sequentially dense in $\mathcal{D}^*(\Omega)$ with respect to the topology $\sigma(\mathcal{D}^*, \mathcal{D})$.
(e) $\mathcal{D}'(\Omega)$ is the completion of $\mathcal{E}(\Omega)$ with regard to $\beta(\mathcal{D}', \mathcal{D})$; $\mathcal{D}^*(\Omega)$ is the completion of $\mathcal{E}(\Omega)$ with regard to $\sigma(\mathcal{D}', \mathcal{D})$.

Proof: (a) The statements about $\tau(\mathcal{D}, \mathcal{D}^*)$ follow from Proposition 2 and Corollary 4, the statements about $\tau(\mathcal{D}, \mathcal{D}')$ from Cor. to Thm. 2.12.3 and Prop. 3.6.4 in [5]. (b) The statements about $\beta(\mathcal{D}', \mathcal{D})$ follow, for example, by combining the assertions from Ex. 3.7.2, Prop. 3.7.6, Ex. 3.9.6, Prop. 3.9.9 and the sentence after Cor. to Prop. 3.9.1 in [5]. The sequential completeness of $\mathcal{D}'(\Omega)$ with $\sigma(\mathcal{D}', \mathcal{D})$ follows from (a) and Corollary 5. (c) This is asserted by Proposition 2 and Corollary 4. (d) The subspace of $\mathcal{D}(\Omega)$ orthogonal to $\mathcal{E}(\Omega)$ is $\{0\}$, thus $\mathcal{E}(\Omega)$ is dense in $\mathcal{D}^*(\Omega)$ [5, Prop. 3.3.3]. The fact that $\mathcal{E}(\Omega)$ is not sequentially dense in $\mathcal{D}^*(\Omega)$ follows from (b), the sequential completeness of $\mathcal{D}'(\Omega)$. (e) By (d), $\mathcal{E}(\Omega)$ is dense in $\mathcal{D}^*(\Omega)$ whose weak topology $\sigma(\mathcal{D}^*, \mathcal{D})$ is complete, whence the second statement. But $\mathcal{E}(\Omega)$ is weakly dense in $\mathcal{D}'(\Omega)$ all the more, hence also dense in $\mathcal{D}'(\Omega)$ with respect to the topology $\beta(\mathcal{D}', \mathcal{D}) = \tau(\mathcal{D}', \mathcal{D})$ [5, Prop. 3.4.3]. □

Properties related to the closed graph theorem. A second important theorem of analysis is the closed graph theorem to which we now turn. Thus let F, G be locally convex spaces and consider the statement

(S2) Every linear map from F into G whose graph is a closed subset of $F \times G$ is continuous.

The classical *closed graph theorem* of Banach [1] says that statement (S2) is true if both F and G are Fréchet spaces. In order to extend this theorem to more general classes of spaces, we might first fix the class on the left hand side, say to the class of barrelled spaces. What spaces then are admitted on the right hand side to make (S2) true? Consider a locally convex space G. A subset $L \subset G'$ is called $\nu(G', G)$-closed, if its intersections $L \cap U$ with all equicontinuous subsets U of G' are $\sigma(G', G)$-closed (in U). The locally convex space G is called *fully complete* or a *Pták space* if all $\nu(G', G)$-closed subspaces of G' are $\sigma(G', G)$-closed. The closed graph theorem of Robertson and Robertson [25] states that (S2) is true if F is barrelled and G a Pták space, see also [5, Thm. 3.17.4].

Let us aim directly at the class of all locally convex spaces G, such that (S2) is true for every barrelled space F. A space G with this property is called *(barrelled)-minimal* or an *infra-(s)-space*. The question whether the spaces

$\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are Pták spaces (or more generally infra-Pták spaces [5, Sect. 3.10]), was settled in the negative by Valdivia [27, 28]. Since every barrelled infra-(s)-space is an infra-Pták space ([16, §34.9.(8)], [21, S. 7.3.9]), it follows that $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are not (barrelled)-minimal; neither with their strong nor their weak topologies [21, Bsp. 7.3.10, Bsp. 7.3.11].

The situation is different with the algebraic dual $\mathcal{D}^*(\Omega)$. Observe first that every subspace of $\mathcal{D}(\Omega)$ is $\sigma(\mathcal{D},\mathcal{D}^*)$-closed (Lemma 1). It follows that $\mathcal{D}^*(\Omega)$ is a Pták space. But much more is actually true: $\mathcal{D}^*(\Omega)$ is (locally convex)-minimal, that is, statement (S2) holds with every locally convex space F when the target space G is $\mathcal{D}^*(\Omega)$. In fact, a locally convex space is (locally convex)-minimal if and only if it is isomorphic to $\mathbb{K}^\Lambda$ for some index set Λ. This is a consequence of the closed graph theorem of Kōmura [17], see e. g. [21, Thm. 7.2.4] for further details.

On the other hand, the space $\mathcal{D}(\Omega)$ with the finest locally convex topology is clearly not (barrelled)-minimal, because $\tau(\mathcal{D},\mathcal{D}')$ is a strictly coarser barrelled topology. We may summarize what has just been deduced as follows:

Proposition 7 *(a) $\mathcal{D}^*(\Omega)$ is a Pták space and (locally convex)-minimal in the topology $\sigma(\mathcal{D}^*,\mathcal{D}) = \tau(\mathcal{D}^*,\mathcal{D}) = \beta(\mathcal{D}^*,\mathcal{D})$.*
(b) $\mathcal{D}'(\Omega)$ is not (barrelled)-minimal, neither with respect to $\sigma(\mathcal{D}',\mathcal{D})$ nor with respect to $\beta(\mathcal{D}',\mathcal{D})$.
(c) $\mathcal{D}(\Omega)$ is not (barrelled)-minimal, neither with respect to $\tau(\mathcal{D},\mathcal{D}')$ nor with respect to $\tau(\mathcal{D},\mathcal{D}^*)$. □

With regard to statement (S2), the situation is much better for the Fréchet space $\mathcal{E}(\Omega)$ and its continuous dual. For example, $\mathcal{E}'(\Omega)$ is a Pták space in the topology $\beta(\mathcal{E}',\mathcal{E})$ [5, Prop. 3.17.6]. Also, both $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are ultrabornological (i. e., inductive limits of an arbitrary family of Fréchet spaces) and (ultrabornological)-minimal, thanks to De Wilde's theory [2]. For more details on the closed graph theorem and its historical aspects we refer to [9, 10, 16, 21].

4 $\mathcal{D}^*(\Omega)$ as a space of generalized functions

Derivation and multiplication. Let Ω be an open subset of $\mathbb{R}^n$. We begin by collecting some positive results about the algebraic dual $\mathcal{D}^*(\Omega)$, to

show that $\mathcal{D}^*(\Omega)$ may serve as a space of generalized functions on Ω. First, an element $S \in \mathcal{D}^*(\Omega)$ can be differentiated and multiplied by smooth functions. The definitions follow the same lines as in [5, Sect. 4.3, Sect. 4.6]. Thus let $p \in \mathbb{N}_0^n$. Then the p-th partial derivative of S is defined as

$$\langle \partial^p S, \varphi \rangle = (-1)^{|p|} \langle S, \partial^p \varphi \rangle$$

and the product of S with a smooth function $\alpha \in \mathcal{E}(\Omega)$ by

$$\langle \alpha S, \varphi \rangle = \langle S, \alpha \varphi \rangle$$

for $\varphi \in \mathcal{D}(\Omega)$. For the one-dimensional case $\Omega = \mathbb{R}$, a repetition of the classical proof shows:

Proposition 8 *The map $\partial : \mathcal{D}^*(\mathbb{R}) \to \mathcal{D}^*(\mathbb{R})$ is surjective; its kernel consists of the one-dimensional subspace of constant functions.*

Proof: Following [5, Sect. 4.3], we denote the image of $\partial : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ by H. A test function χ belongs to H if and only if its integral vanishes. Taking an arbitrary test function φ_0 with integral 1, every $\varphi \in \mathcal{D}(\mathbb{R})$ can be written as

$$\varphi = \lambda \varphi_0 + \chi$$

with $\chi = \partial \psi \in H$ and $\lambda = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x$. Let $T \in \mathcal{D}^*(\mathbb{R})$ and $\partial T = 0$. Then

$$\langle T, \varphi \rangle = \lambda \langle T, \varphi_0 \rangle - \langle \partial T, \psi \rangle = \langle T, \varphi_0 \rangle \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x.$$

This shows that the action of T is given by the constant function $\langle T, \varphi_0 \rangle$. On the other hand, we have the direct sum decomposition

$$\mathcal{D}(\mathbb{R}) = \mathbb{C}\,\varphi_0 \oplus H,$$

and $-\partial : \mathcal{D}(\mathbb{R}) \to H$ is injective. Thus given $S \in \mathcal{D}^*(\mathbb{R})$, the element $T \in \mathcal{D}^*(\mathbb{R})$,

$$\langle T, \varphi \rangle = -\langle S, \psi \rangle$$

is well defined (where ψ has the same meaning as above), and clearly $\partial T = S$. □

Let $h \in \mathbb{R}^n$. The translate of a function $f \in \mathcal{E}(\mathbb{R}^n)$ is defined by $(\tau_h f)(x) = f(x - h)$. The translate of an element T of $\mathcal{D}^*(\mathbb{R}^n)$ can be defined by

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle$$

as in the case of distributions [5, Def. 4.3.2]. The map $h \to \tau_h T$ from $\mathbb{R}^n$ to $\mathcal{D}^*(\mathbb{R}^n)$ is continuously differentiable when $\mathcal{D}^*(\Omega)$ is equipped with the topology $\sigma(\mathcal{D}^*, \mathcal{D})$ and

$$\frac{\partial}{\partial h_j}(\tau_h T) = -\tau_h(\partial_j T).$$

Just as in [5, Prop. 4.3.6], we have that an element $T \in \mathcal{D}^*(\mathbb{R}^n)$ is *independent of the variable* x_j if and only if $\tau_h T = T$ for all vectors $h \in \mathbb{R}^n$ parallel to the x_j-axis, if and only if $\partial_j T = 0$.

Supports and restrictions. We now turn to the sheaf properties of $\mathcal{D}^*(\Omega)$. First, if U is an open subset of Ω every function $\varphi \in \mathcal{D}(U)$ can be considered as an element of $\mathcal{D}(\Omega)$, extending it by 0 outside U. Thus an element $T \in \mathcal{D}^*(\Omega)$ can be restricted to U by the prescription

$$\langle T|U, \varphi\rangle = \langle T, \varphi\rangle$$

for $\varphi \in \mathcal{D}(U)$. Clearly, if $V \subset U$ then $T|V = (T|U)|V$. We say that T vanishes on U if $T|U = 0$. Given an open cover $(\Omega_\iota)_{\iota\in I}$ of Ω we have that T vanishes on Ω if and only if all its restrictions to Ω_ι vanish for every $\iota \in I$. Indeed, there is a locally finite, infinitely differentiable partition of unity $(\alpha_\iota)_{\iota\in I}$ subordinated to the cover $(\Omega_\iota)_{\iota\in I}$ [5, Thm. 2.12.4]. For $\varphi \in \mathcal{D}(\Omega)$ we thus have

$$\langle T, \varphi\rangle = \sum_{\iota\in I}\langle T, \alpha_\iota\varphi\rangle = 0$$

since the sum contains only finitely many terms when φ is fixed. In the same vein, given a family of elements $T_\iota \in \mathcal{D}^*(\Omega_\iota)$, $\iota \in I$, such that $T_\iota = T_\kappa$ on each non-empty common domain $\Omega_\iota \cap \Omega_\kappa$, there is a unique element $T \in \mathcal{D}^*(\Omega)$ such that $T|\Omega_\iota = T_\iota$ for every $\iota \in I$. The proof of this fact is the same as in [5, Prop. 4.2.4]; actually shorter since the continuity argument is not needed. We have proven:

Proposition 9 *The assignment $\Omega \to \mathcal{D}^*(\Omega)$ defines a sheaf of locally convex spaces on $\mathbb{R}^n$.* □

In particular, the support of an element $T \in \mathcal{D}^*(\Omega)$ is well defined as the complement of the largest open subset of Ω on which it vanishes.

Here comes the first major difference of the behavior of $\mathcal{D}^*$ as compared to $\mathcal{D}'$. As mentioned in Section 2, the elements of $\mathcal{E}'(\Omega)$ can be identified with the

compactly supported distributions. This is no longer the case in the setting of the algebraic duals: $\mathcal{D}(\Omega)$ is not a dense subspace of $\mathcal{E}(\Omega)$ with respect to the finest locally convex topology $\tau(\mathcal{E}, \mathcal{E}^*)$, but rather a closed subspace (Lemma 1); hence the transpose of this imbedding is not an injective map from $\mathcal{E}^*(\Omega)$ to $\mathcal{D}^*(\Omega)$. On the contrary, we have an injection in the reverse direction. To see this, let N be an algebraic supplement of $\mathcal{D}(\Omega)$ in $\mathcal{E}(\Omega)$. The map

$$i : \mathcal{D}^*(\Omega) \to \mathcal{E}^*(\Omega), \quad \langle i(T), \varphi\rangle = \langle T, \psi\rangle \tag{2}$$

where $T \in \mathcal{D}^*(\Omega)$ and $\varphi = \psi + \chi$ with $\psi \in \mathcal{D}(\Omega)$ and $\chi \in N$ is clearly linear and injective. This way $\mathcal{D}^*(\Omega)$ becomes a subspace of $\mathcal{E}^*(\Omega)$, and membership in $\mathcal{E}^*(\Omega)$ does not correspond to any support property. There are many elements of $\mathcal{E}^*(\Omega)$ with the same action as T on $\mathcal{D}(\Omega)$, namely all those of the form $\varphi \to \langle T, \psi\rangle + \langle T', \chi\rangle$ where T' is some linear functional on N.

A similar situation arises with respect to the spaces of distributions of finite order. Let $\mathcal{D}^m(\Omega), m \in \mathbb{N}_0$, denote the space of m-times continuously differentiable functions with compact support. Its continuous dual $\mathcal{D}'^m(\Omega)$ is the space of distributions of order (at most) m and is a subspace of $\mathcal{D}'(\Omega)$. Again, in the setting of algebraic duals, the injections are reversed: Letting N^m be an algebraic supplement of $\mathcal{D}(\Omega)$ in $\mathcal{D}^m(\Omega)$, the same reasoning as in (2) leads to a linear injection of $\mathcal{D}^*(\Omega)$ in $\mathcal{D}^{*m}(\Omega)$. Thus the notion of *order* has no meaning for the elements of $\mathcal{D}^*(\Omega)$. Indeed, we will shortly exhibit elements that do not arise as distributions of locally finite order.

Example 10 *Let M be the subspace of $\mathcal{D}(\mathbb{R})$ of test functions whose sequence of derivatives at zero is summable:*

$$M = \{\psi \in \mathcal{D}(\mathbb{R}) : \sum_{k=0}^{\infty} |\partial^k \psi(0)| < \infty\},$$

and let N be an algebraic supplement of M in $\mathcal{D}(\mathbb{R})$. The prescription

$$\langle T, \varphi\rangle = \sum_{k=0}^{\infty} \partial^k \psi(0)$$

where $\varphi = \psi + \chi$ with $\psi \in M, \chi \in N$ defines an element $T \in \mathcal{D}^*(\mathbb{R})$. Involving infinitely many derivatives at zero, T is not a continuous functional on $\mathcal{D}(\mathbb{R})$ with respect to $\mathcal{T}_D$, thus does not belong to $\mathcal{D}'(\mathbb{R})$.

Actually, the simple algebraic argument in (2) can be generalized to show that the spaces of Gevrey ultradistributions are also contained as subspaces of $\mathcal{D}^*(\Omega)$. Thus we have the somewhat curious sequence of inclusions ($\sigma > 1$)

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{D}'_\sigma(\Omega) \subset \mathcal{D}^*_\sigma(\Omega) \subset \mathcal{D}^*(\Omega) \subset \mathcal{E}^*(\Omega).$$

Example 10 is also of interest from the viewpoint of supports: the support of the distribution T defined there is $\{0\}$. Indeed, if $\varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, then φ belongs to M and $\langle T, \varphi\rangle = 0$. We see that an element of $\mathcal{D}(\mathbb{R})$ whose support is $\{0\}$ need not be a finite linear combination of the Dirac measure and its derivatives (as opposed to the case of distributions, [5, Prop. 4.4.5]). Here is another example of this phenomenon.

Example 11 *Fix an element φ_0 of $\mathcal{D}(\mathbb{R})$ such that $\varphi_0(0) = 1$ and $\partial^k\varphi_0(0) = 0$ for all $k \geq 1$. Let H be an algebraic supplement of the one-dimensional space $\mathbb{C}\,\varphi_0$ in $\mathcal{D}(\mathbb{R})$. Define a linear form S on $\mathcal{D}(\mathbb{R})$ by*

$$\langle S, \varphi\rangle = \lambda,$$

where $\varphi = \lambda\varphi_0 + \chi$ with $\lambda \in \mathbb{K}$ and $\chi \in H$. Then S is not a finite linear combination of the Dirac measure δ and its derivatives. Indeed, assume to the contrary that $S = \sum_{p=0}^m a_p \partial^p \delta$ for some $m \in \mathbb{N}$ and certain coefficients a_p. Letting $\varphi = \lambda\varphi_0 + \chi$ with $\chi \in H$, we would have that

$$\langle \sum_{p=0}^m a_p \partial^p \delta, \varphi\rangle = \sum_{p=0}^m (-1)^p a_p \partial^p \varphi(0) = \lambda + \sum_{p=0}^m (-1)^p a_p \partial^p \chi(0). \tag{3}$$

If this expression represented $\langle S, \varphi\rangle$ it should equal λ, for arbitrary choices of $\chi \in H$. This is not the case, because one can always find elements χ of $\mathcal{D}(\mathbb{R})$ which are not multiples of φ_0 such that the sum on the right hand side of (3) does not vanish.

Convolutions and tensor product. We now arrive at a more severe failure of $\mathcal{D}(\mathbb{R})$, and that is the failure of convolutions to regularize.

Lemma 12 *Let φ be a nonzero element of $\mathcal{D}(\mathbb{R})$. Then the family of translates $(\tau_h\varphi)_{h\in\mathbb{R}}$ is linearly independent in $\mathcal{D}(\mathbb{R})$.*

Proof: Assume that

$$\sum_{p=0}^{m} a_p \tau_{h_p} \varphi(x) \equiv 0 \text{ on } \mathbb{R}$$

for certain $m \in \mathbb{N}, h_p \in \mathbb{R}$ and $a_p \in \mathbb{C}$. Taking the Fourier transform of this equation, we have that

$$\Big(\sum_{p=0}^{m} a_p \mathrm{e}^{-\mathrm{i} h_p \xi}\Big) \mathcal{F}\varphi(\xi) \equiv 0 \text{ on } \mathbb{R}.$$

Since both factors can be extended as entire functions of ξ to the complex plane and the ring of holomorphic functions has no zero divisors, it follows that

$$\sum_{p=0}^{m} a_p \mathrm{e}^{-\mathrm{i} h_p \zeta} \equiv 0 \text{ on } \mathbb{C}.$$

But exponentials of different phase are linearly independent, thus all coefficients a_p necessarily vanish. $\square$

Definition 13 *Let $S \in \mathcal{D}^*(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n), x \in \mathbb{R}^n$. The convolution of S and φ at the point x is defined as $S * \varphi(x) = \langle S, \tau_x \check{\varphi} \rangle$.*

Here $\check{\varphi}(y) = \varphi(-y)$; in abusive notation involving the independent variable inside the duality brackets, the definition may become intuitively clearer:

$$S * \varphi(x) = \langle S(y), \varphi(x-y) \rangle.$$

As in the case of distributions, the convolution with a test function yields a function from $\mathbb{R}^n$ to $\mathbb{K}$. However, it need no longer be smooth, not even continuous.

Example 14 *Let φ be a nonzero element of $\mathcal{D}(\mathbb{R})$. Consider the sequence $h_n = 1/n, h_0 = 0$ in $\mathbb{R}$. By Lemma 4 the sequence of translates $(\tau_{h_n} \check{\varphi})_{n \in \mathbb{N}_0}$ is a linearly independent subset of $\mathcal{D}(\mathbb{R})$. Denote by M its span and by N an algebraic supplement of M in $\mathcal{D}(\mathbb{R})$. Define an element S of $\mathcal{D}^*(\mathbb{R})$ by*

$$\langle S, \check{\varphi} \rangle = 0, \quad \langle S, \tau_{h_n} \check{\varphi} \rangle = n \text{ for } n \geq 1, \quad \langle S, \chi \rangle = 0 \text{ for } \chi \in N.$$

Then obviously

$$S * \varphi\left(\tfrac{1}{n}\right) = \langle S, \tau_{h_n} \check{\varphi} \rangle = n, \quad S * \varphi\,(0) = \langle S, \check{\varphi} \rangle = 0$$

so that the function $x \to S * \varphi(x)$ is discontinuous at zero.

When S belongs to $\mathcal{D}'(\mathbb{R}^n)$ and φ to $\mathcal{D}(\mathbb{R}^n)$, the map $x \to S * \varphi(x)$ is smooth. As can be seen from [5, Prop. 4.10.1], the fact that S is a *continuous* functional on $\mathcal{D}(\mathbb{R}^n)$ with respect to the topology $\mathcal{T}_{\mathcal{D}}$ is at the core of the proof of this property.

Similar difficulties arise when one wants to define the tensor product in the setting of the algebraic duals. Thus let Ξ be an open subset of $\mathbb{R}^k$ and H an open subset of $\mathbb{R}^l$. Let $S \in \mathcal{D}'(\Xi), T \in \mathcal{D}'(\mathrm{H})$. Given $\chi \in \mathcal{D}(\Xi \times \mathrm{H})$, the map (notation as explained after Definition 13)

$$x \to \langle T(y), \chi(x,y) \rangle \tag{4}$$

belongs to $\mathcal{D}(\Xi)$ [5, Lemma 4.8.2]. Thus one may define the tensor product of S and T by

$$\langle S \otimes T, \chi \rangle = \langle S(x), \langle T(y), \chi(x,y) \rangle \rangle$$

and it belongs to $\mathcal{D}'(\Xi \times \mathrm{H})$ [5, Lemma 4.8.3]. Alternatively, let us denote by $\mathcal{D}(\Xi) \otimes \mathcal{D}(\mathrm{H})$ the span in $\mathcal{D}(\Xi \times \mathrm{H})$ of elements of the form $\chi(x,y) = (\varphi \otimes \psi)(x,y) = \varphi(x)\psi(y)$. Then $\mathcal{D}(\Xi) \otimes \mathcal{D}(\mathrm{H})$ is dense in $\mathcal{D}(\Xi \times \mathrm{H})$ [5, Prop. 4.8.1], and $S \otimes T$ turns out to be the unique distribution $R \in \mathcal{D}'(\Xi \times \mathrm{H})$ such that

$$\langle R, \varphi \otimes \psi \rangle = \langle S, \varphi \rangle \langle T, \psi \rangle,$$

see [5, Def. 4.8.1]. Both approaches fail in the setting of the algebraic duals.

Example 15 *Similar to Example 14, fix an element* $\chi \in \mathcal{D}(\mathbb{R} \times \mathbb{R})$ *such that* $\chi(x,y) = 1$ *when* $\max(|x|,|y|) \leq 1$. *The function* $(x,y) \to \mathrm{e}^{\mathrm{i}xy}\chi(x,y)$ *belongs to* $\mathcal{D}(\mathbb{R} \times \mathbb{R})$ *as well, and the family of functions*

$$y \to \chi(0,y), \quad y \to \mathrm{e}^{\mathrm{i}\frac{1}{n}y}\chi(\tfrac{1}{n}, y), \ n \in \mathbb{N},$$

is a linearly independent subset of $\mathcal{D}(\mathbb{R})$. *Indeed, for* $|y| \leq 1$ *we actually have that* $\mathrm{e}^{\mathrm{i}\frac{1}{n}y}\chi(\frac{1}{n}, y) \equiv \mathrm{e}^{\mathrm{i}\frac{1}{n}y}$, *and this family is linearly independent, as noted in the proof of Lemma 12. Now define* $T \in \mathcal{D}^*(\mathbb{R})$ *by*

$$\langle T(y), \chi(0,y) \rangle = 0, \quad \langle T(y), \mathrm{e}^{\mathrm{i}\frac{1}{n}y}\chi(\tfrac{1}{n}, y) \rangle = n$$

for $n \in \mathbb{N}$ *and extend it by zero on an algebraic supplement of the span of this family of functions. Then the map required in (4), with* $\mathrm{e}^{\mathrm{i}xy}\chi(x,y)$ *in place of* $\chi(x,y)$,

$$f : x \to \langle T(y), \mathrm{e}^{\mathrm{i}xy}\chi(x,y) \rangle$$

is again discontinuous at $x = 0 : f(\frac{1}{n}) = n, f(0) = 0$.

As an alternative approach to defining the tensor product of elements $S \in \mathcal{D}'(\Xi), T \in \mathcal{D}'(\mathrm{H})$ one could start with the subspace $\mathcal{D}(\Xi) \otimes \mathcal{D}(\mathrm{H})$ of $\mathcal{D}(\Xi \times \mathrm{H})$ and define

$$\langle R, \sum_{i,j} a_{ij} \varphi_i \otimes \psi_j \rangle = \sum_{i,j} a_{ij} \langle S, \varphi_i \rangle \langle T, \psi_j \rangle.$$

But – as every subspace of $\mathcal{D}(\Xi \times \mathrm{H})$ – the subspace $\mathcal{D}(\Xi) \otimes \mathcal{D}(\mathrm{H})$ is closed for the finest locally convex topology. Thus there are many extensions of R to all of $\mathcal{D}(\Xi \times \mathrm{H})$, and so it remains ambiguous how to define $S \otimes T$ in this way (and to keep control about, e. g., consistency with classical definitions).

As presented in [5, Sect. 4.9], the definition of the convolution of two distributions (with supports in favorable position) is based on the tensor product of distributions. From what has just been said, it is impossible to give a meaning to the convolution of two elements of $\mathcal{D}^*(\mathbb{R}^n)$ along these lines. However, one could try to define the convolution of an element S of $\mathcal{D}^*(\mathbb{R}^n)$ with a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ as follows. Recall that the inflection of T is defined by

$$\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle.$$

Given $\psi \in \mathcal{D}(\mathbb{R}^n)$, the convolution of the distribution $\check{T}$ with ψ is a well defined smooth function, that is, $\check{T} * \psi$ belongs to $\mathcal{E}(\mathbb{R}^n)$ if $T \in \mathcal{D}'(\mathbb{R}^n)$ and to $\mathcal{D}(\mathbb{R}^n)$ if $T \in \mathcal{E}'(\mathbb{R}^n)$ [5, Prop. 4.10.1, Prop. 4.9.2].

Definition 16 *Let $S \in \mathcal{E}^*(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$. Then the convolution $S \star T$ is defined as an element of $\mathcal{D}^*(\mathbb{R}^n)$ by*

$$\langle S \star T, \psi \rangle = \langle S, \check{T} * \psi \rangle \tag{5}$$

for $\psi \in \mathcal{D}(\mathbb{R}^n)$.

Remark 17 *By what has been said just before Definition 16, the right hand side of formula (5) makes sense if $S \in \mathcal{D}^*(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$. On the other hand, $\mathcal{D}^*(\mathbb{R}^n)$ is imbedded in $\mathcal{E}^*(\mathbb{R}^n)$ by means of the injection i given in (2), and so one may also consider the convolution $i(S) \star T$ according to Definition 16. Due to the construction of the injection i, the two formulas give rise to the same result when $T \in \mathcal{E}'(\mathbb{R}^n)$, because $\langle i(S), \varphi \rangle = \langle S, \varphi \rangle$ when φ has compact support.*

However, other than that not much can be said about consistency with classically defined convolutions. In fact, Definition 16 is not consistent with Definition 13 when T is a test function itself.

Example 18 *We continue with Example 14. First observe that if $x \in \mathbb{R}$ is not one of the members of the sequence $h_n, n \in \mathbb{N}_0$, then $\tau_x \check{\varphi}$ does not lie in its span M (Lemma 4). In addition, if we take φ with support in the half line $(-\infty, 0]$, then the function $\varphi * \check{\varphi}$ does not belong to M either (because it is symmetric, while all members of M vanish on $(-\infty, 0]$). Thus we may modify the direct sum composition $\mathcal{D}(\mathbb{R}) = M \oplus N$ as follows: we adjoin $\varphi * \check{\varphi}$ to M and set up the algebraic complement N in such a way that each $\tau_x \check{\varphi}$ belongs to N when x is not equal to one of the members h_n. We also modify the definition of $S \in \mathcal{D}^*(\mathbb{R})$ as follows:*

$$\langle S, \check{\varphi} \rangle = 0, \quad \langle S, \tau_{h_n} \check{\varphi} \rangle = n \text{ for } n \geq 1, \quad \langle S, \varphi * \check{\varphi} \rangle = 1, \quad \langle S, \chi \rangle = 0 \text{ for } \chi \in N.$$

As a consequence, we have that $S * \varphi(\frac{1}{n}) = n$, while $S * \varphi(x) = 0$ for all other $x \in \mathbb{R}$; in particular, the function $x \to S * \varphi(x)$ is zero almost everywhere. If we choose to view it as an element of $\mathcal{D}^*(\mathbb{R})$ by means of the imbedding of $\mathcal{D}'(\mathbb{R})$, it is the zero element. On the other hand, the convolution of S and φ according to Definition 16 is given by $\langle S \star \varphi, \psi \rangle = \langle S, \check{\varphi} * \psi \rangle$ for $\psi \in \mathcal{D}(\mathbb{R})$. Taking in particular $\psi = \varphi$ we have by construction $\langle S \star \varphi, \varphi \rangle = 1$, clearly inconsistent with Definition 13 according to which $\langle S * \varphi, \varphi \rangle = \int S * \varphi(x)\, \varphi(x)\, \mathrm{d}x = 0$.

Thus serious problems arise when one attempts to define convolutions in $\mathcal{D}^*(\mathbb{R}^n)$. But it is worthwhile to note that the convolution introduced in Definition 16 behaves well with respect to derivatives. If S, T are as in Definition 16 or in Remark 17 then

$$\partial_j (S \star T) = (\partial_j S) \star T = S \star (\partial_j T). \tag{6}$$

This follows immediately from formula (5), the corresponding property of convolution of distributions, and the definition of partial derivatives on $\mathcal{D}^*(\mathbb{R}^n)$.

5 Solving linear partial differential equations in $\mathcal{D}^*(\Omega)$

We begin this section by an elementary observation on surjections of algebraic duals.

Proposition 19 *Let E be a vector space and $P : E^* \to E^*$ a linear mapping. Then P is surjective if and only if its transpose ${}^tP : E \to E$ is injective.*

Proof: Assume first that ${}^tP$ is injective. Let N be an algebraic supplement of ${}^tP(E)$ in E. Given $y^* \in E^*$, define an element $x^* \in E^*$ by $\langle x^*, z\rangle = \langle y^*, y\rangle$ for $z = {}^tP(y) \in {}^tp(E)$, $\langle x^*, z\rangle = 0$ for $z \in N$. Since ${}^tP$ is injective, the element x^* of E^* is well defined, and clearly $\langle P(x^*), x\rangle = \langle x^*, {}^tP(x)\rangle = \langle y^*, x\rangle$ for $x \in E$. Conversely, assume that P is surjective and let ${}^tP(y) = 0$. Then $\langle y^*, y\rangle = \langle x^*, {}^tP(y)\rangle = 0$ for all $y^* \in E^*$, taking $x^* \in E^*$ such that $y^* = P(x^*)$. Thus $y = 0$. $\square$

We shall now explore what this purely algebraic result (no continuity is needed) can or cannot say about solvability of partial differential equations. Thus let Ω be an open subset of $\mathbb{R}^n$ and

$$P(x, \partial) = \sum_{|p|\le m} a_p(x)\, \partial^p$$

be a linear partial differential operator with smooth coefficients $a_p \in \mathcal{E}(\Omega)$. Viewing $P(x, \partial)$ as an operator acting on $\mathcal{D}^*(\Omega)$, its transpose ${}^tP(x, \partial)$ is given by

$${}^tP(x, \partial)\, \varphi(x) = \sum_{|p|\le m} (-1)^{|p|}\, \partial^p\, (a_p(x)\varphi(x))\ .$$

By the proposition above, if ${}^tP(x, \partial) : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is injective, then the equation

$$P(x, \partial)U = F \tag{7}$$

has a solution $U \in \mathcal{D}^*(\Omega)$ for whatever $F \in \mathcal{D}^*(\Omega)$. This line of arguments to establish solvability was first used by Todorov [26] in the context of a factor space of the space of nonstandard internal smooth functions. The $\mathcal{D}^*$-version was elaborated jointly with him in an unpublished manuscript [22].

Example 20 *(a) If Ω is an arbitrary open subset of $\mathbb{R}^n$ and $P = P(\partial)$ is a linear partial differential operator with constant coefficients, then ${}^tP : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is injective. To see this, it suffices to take the Fourier transform of the equation ${}^tP(\partial)\varphi = 0$ and invoke the analyticity of $\mathcal{F}\varphi$.*
(b) If Ω is an arbitrary open subset of $\mathbb{R}^n$ and $P = P(x, \partial)$ is a linear partial differential operator of constant strength with analytic coefficients then ${}^tP(x, \partial) : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is injective (see the discussion before Thm. 13.5.2 in [4]). This is true, in particular, when $P(x, \partial)$ is an elliptic operator with analytic coefficients.

(c) If Ω is an arbitrary open subset of $\mathbb{R}^3$ and P the Lewy operator

$$P(x,\partial) = -\frac{\partial}{\partial x_1} - \mathrm{i}\,\frac{\partial}{\partial y_1} + 2\mathrm{i}(x_1 + iy_1)\,\frac{\partial}{\partial x_2} \tag{8}$$

then ${}^tP(x,\partial) = -P(x,\partial) : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is injective. This can be seen, e. g., by the following elementary argument. Perform a partial Fourier transform of the equation $-P\varphi(x_1, y_1, x_2) \equiv 0$ with respect to the variable x_2. Then $(\mathcal{F}_{x_2\to z_2}\varphi)(x_1, x_2, z_2) = \psi(x_1, x_2, z_2)$ is an entire function of z_2 at fixed (x_1, y_1). Viewing (x_1, y_1) as the complex variable $z_1 = x_1 + \mathrm{i}y_1$, we see that ψ satisfies the equation

$$\Big(\frac{\partial}{\partial \overline{z_1}} + z_1 z_2\Big)\psi(z_1, z_2) \equiv 0. \tag{9}$$

Setting $z_2 = 0$, (9) implies that the function $z_1 \to \psi(z_1, 0)$ is analytic; having compact support, it necessarily vanishes identically. Successively differentiating (9) with respect to z_2 and setting $z_2 = 0$, we observe that $\partial^k_{z_2}\psi(z_1, 0) = 0$ for all $z_1 \in \mathbb{C}, k \in \mathbb{N}_0$. Recalling the analyticity of $z_2 \to \psi(z_1, z_2)$, it follows that ψ, and hence φ, vanishes identically.
(d) Of course, there are many operators which are not injective on $\mathcal{D}(\Omega)$. For example, the operator $(1 - x^2)^2\partial_x + 2x : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is not injective; the smooth function which equals $\exp(-1/(1 - x^2))$ for $|x| < 1$ and 0 otherwise is in its kernel. The operator $x_2\partial_{x_1} - x_1\partial_{x_2} : \mathcal{D}(\mathbb{R}^2) \to \mathcal{D}(\mathbb{R}^2)$ is not injective; all rotationally invariant functions belong to its kernel. There is even a fourth order elliptic operator with smooth coefficients which is not injective from $\mathcal{D}(\mathbb{R}^3) \to \mathcal{D}(\mathbb{R}^3)$ [4, Thm. 13.6.15].

We now discuss some special cases of the examples just mentioned in more detail. The general solvability assertion in $\mathcal{D}^*(\Omega), \Omega \subset \mathbb{R}^3$, in Example 20 (c) is curious in view of the fact that the operator $P(x,\partial)$ from (8) provided the first example, due to Lewy [18], of an operator with smooth coefficients which is not locally solvable in the sense of distributions. That is, there exist smooth functions $F \in \mathcal{E}(\mathbb{R}^3)$ such that the equation $P(x,\partial)U = F$ does not have a solution in $\mathcal{D}'(\Omega)$ for whatever open subset $\Omega \subset \mathbb{R}^n$. In contrast, it does have solutions in $\mathcal{D}^*(\Omega)$.

Example 20 (a) and Proposition 19 immediately imply the following assertion.

Corollary 21 *Let Ω be an arbitrary open subset of $\mathbb{R}^n$, $P = P(\partial)$ a nonzero linear partial differential operator with constant coefficients. Then $P(\partial) : \mathcal{D}^*(\Omega) \to \mathcal{D}^*(\Omega)$ is surjective.* $\square$

This is in contrast with the classical situation that the equation $P(\partial)U = F$ has a solution $U \in \mathcal{D}'(\Omega)$ for every $F \in \mathcal{D}'(\Omega)$ if and only if Ω is P-*convex for supports as well as singular supports* [4, Cor. 10.7.10], while the equation $P(\partial)U = F$ has a solution $U \in \mathcal{D}'(\Omega)$ for every $F \in \mathcal{E}(\Omega)$ if and only if Ω is P-*convex for supports* [4, Thm. 10.6.6, Cor. 10.6.8]. In this context, we can observe that the property of *hypoellipticity* may be lost when admitting solutions in $\mathcal{D}^*(\Omega)$. Recall that an operator $P(\partial)$ is hypoelliptic, if for whatever open subset $\Omega \subset \mathbb{R}^n$ and $U \in \mathcal{D}'(\Omega)$, $P(\partial)U \in \mathcal{E}(\Omega)$ implies $U \in \mathcal{E}(\Omega)$. Let $P(\partial)$ be a hypoelliptic operator which is not elliptic. Then there exists an open subset Ω of $\mathbb{R}^n$ which is not P-convex for supports [4, Cor. 10.8.2]. Thus there is $F \in \mathcal{E}(\Omega)$ such that the equation $P(\partial)U = F$ has no solution $U \in \mathcal{D}'(\Omega)$. However, by the corollary above, it does have a solution $U \in \mathcal{D}^*(\Omega)$. By what has just been said, this solution does not belong to $\mathcal{E}(\Omega)$; thus solutions in $\mathcal{D}^*(\Omega)$ of hypoelliptic operators with smooth right hand side need not be smooth.

A *fundamental solution* of a constant coefficient partial differential operator $P(\partial)$ is an element S of $\mathcal{D}^*(\mathbb{R}^n)$ such that $P(\partial)S = \delta$, the Dirac measure. Corollary 21 implies, in particular, that every nonzero constant coefficient partial differential operator possesses a fundamental solution in $\mathcal{D}^*(\mathbb{R}^n)$. Due to the theorem of Malgrange and Ehrenpreis [3, 20], every nonzero constant coefficient partial differential operator actually has a fundamental solution in $\mathcal{D}'(\mathbb{R}^n)$. We refer to [23, 24] for an elegant explicit construction and a historical survey. Proposition 19 is just the simple portion of the Malgrange-Ehrenpreis theorem, the difficult part of course being to prove the continuity of the functional defined on the range of ${}^tP$.

Proposition 22 *(a) Let S be a fundamental solution of $P(\partial)$ in $\mathcal{D}^*(\mathbb{R}^n)$ and let $F \in \mathcal{E}'(\mathbb{R}^n)$. Then the element $U = S \star F \in \mathcal{D}^*(\mathbb{R}^n)$ is a solution of the equation $P(\partial)U = F$ in $\mathcal{D}^*(\mathbb{R}^n)$.*
(b) Let T be a fundamental solution of $P(\partial)$ in $\mathcal{D}'(\mathbb{R}^n)$ and let $G \in \mathcal{E}^*(\mathbb{R}^n)$. Then the element $V = G \star T \in \mathcal{D}^*(\mathbb{R}^n)$ is a solution of the equation $P(\partial)V = G$ in $\mathcal{D}^*(\mathbb{R}^n)$.

Proof: By Remark 17, both U and V are well defined elements of $\mathcal{D}^*(\mathbb{R}^n)$ according to formula (5). Using (6) we have in case (a) that

$$P(\partial)U = (P(\partial)S) \star F = \delta \star F = F.$$

The latter equality follows from

$$\langle \delta \star F, \psi \rangle = \langle \delta, \check{F} * \psi \rangle = (\check{F} * \psi)(0) = \langle F, \psi \rangle.$$

for $\psi \in \mathcal{D}(\mathbb{R}^n)$. In case (b), we have that

$$P(\partial)V = G \star (P(\partial)T) = G \star \delta = G,$$

using that $\langle G \star \delta, \psi \rangle = \langle G, \check{\delta} * \psi \rangle = \langle G, \psi \rangle$. $\square$

Thus the simple tool of Proposition 19 allows to solve constant coefficient partial differential equations in $\mathcal{D}^*(\mathbb{R}^n)$, though not much can be inferred about the properties of these solutions in general.

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